AN INTERPRETATION OF BAROTROPIC INSTABILITY IN TERMS OF THE FREQUENCY OF MOMENTUM CONVERGENCE [1]

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ABSTRACT

The comparative study of simplified versions of the equations governing barotropic flow in 1) the f-plane, 2) the β-plane, 3) the spherical surface, leads to a new interpretation of the stabilizing effect of the β-term on the long waves. The relevant observation is that although in the long-wave region the momentum convergence has a large amplitude, its time frequency is also large, so that no significant energy conversion can be performed.

1. INTRODUCTION

One of the most important properties of spectral forms of the dynamical equations is that severely truncated versions of the equations possess the same quadratic invariants as the full set of equations. This fact lends support to the idea, proposed by Lorenz [5], that the dynamical equations may be stripped of a large amount of detail without sacrificing their essential physical content. Thus, the truncated equations, or "low-order" system, may be used to represent the nonlinear effects inherent in the full dynamical equations in a very simple manner.

This technique has been applied by many authors (Platzman and Baer [8], Wiin-Nielsen [10], Bryan [1], Eliasen [2]) to describe the exchange of energy between the eddies and the zonal flow. These authors have also developed and illustrated the concept of available *kinetic* energy, which depends on the conservation of the total momentum of the flow.

There is one aspect of this problem that has not received any attention in the literature; and that is the physical interpretation of barotropic instability in the "low-order" system. Bryan [1] recognized the importance of the differential effect of the β-term in determining the characteristics of the barotropic energy exchange, but did not explicitly connect it with barotropic instability.

The purpose of this paper is to provide this connection between barotropic instability and the differential effect of the β-term; and for this purpose three models of barotropic flow will be studied in a comparative way.

The first model was presented by Lorenz [5], and considers barotropic motion on a flat earth where the Coriolis parameter is constant (the "f-plane") the second is the extension of Lorenz's model to the "β-plane"; the third is an equivalent three-component system in spherical harmonics which accounts for the sphericity of the earth. Because of the geometry of the first two models, the relevant functions to be used in the spectral representation of the dynamical equations are trigonometric functions of x, y, where x is distance measured in an east-west direction, y is distance measured in a north-south direction.

2. MODEL 1—MOTION ON THE "f-PLANE"

The equation governing the first model is

$$\frac{\partial}{\partial t}(\nabla^2\psi)+\mathbf{k}\times\nabla\psi\cdot\nabla(\nabla^2\psi)=0, \tag{1}$$

where ψ is a stream function and $\mathbf{k}$ is the vertical unit vector. Lorenz shows that by considering flows which are doubly periodic in x and y equation (1) may be transformed into spectral form in terms of trigonometric functions of the form $e^{i(mkx+nly)}$ where m, n are integers; and $L_x=2\pi/k$, $L_y=2\pi/l$ define the fundamental region. Further, by truncating the representation he shows that the minimum system capable of reproducing the nonlinear effect of the advection term in (1) is given by the following equations:

$$\nabla^2\psi=A\cos ly+F\cos kx+2G\sin ly\sin kx,$$
$$\psi=-\frac{A}{l^2}\cos ly-\frac{F}{k^2}\cos kx-\frac{2G}{k^2+l^2}\sin ly\sin kx. \tag{2}$$

Then the harmonic tendency equations obtained by substituting (2) in (1) are

$$\frac{dA}{dt}=-\frac{1}{\alpha(\alpha^2+1)}FG,$$
$$\frac{dF}{dt}=\frac{\alpha^3}{(\alpha^2+1)}AG,$$
$$\frac{dG}{dt}=-\frac{\alpha^2-1}{2\alpha}AF, \tag{3}$$

where $\alpha=k/l$.

The expressions for the horizontally averaged kinetic energy and square vorticity are given by

$$\overline{E}=\frac{1}{4}\left(\frac{A^2}{l^2}+\frac{F^2}{k^2}+\frac{2G^2}{k^2+l^2}\right),\ \overline{V}^2=\frac{1}{2}(A^2+F^2+2G^2). \tag{4}$$

They are both conserved under this truncation as may be verified by using equations (3).

The differential equations (3) can be solved analytically and the solution can be expressed in terms of elliptic

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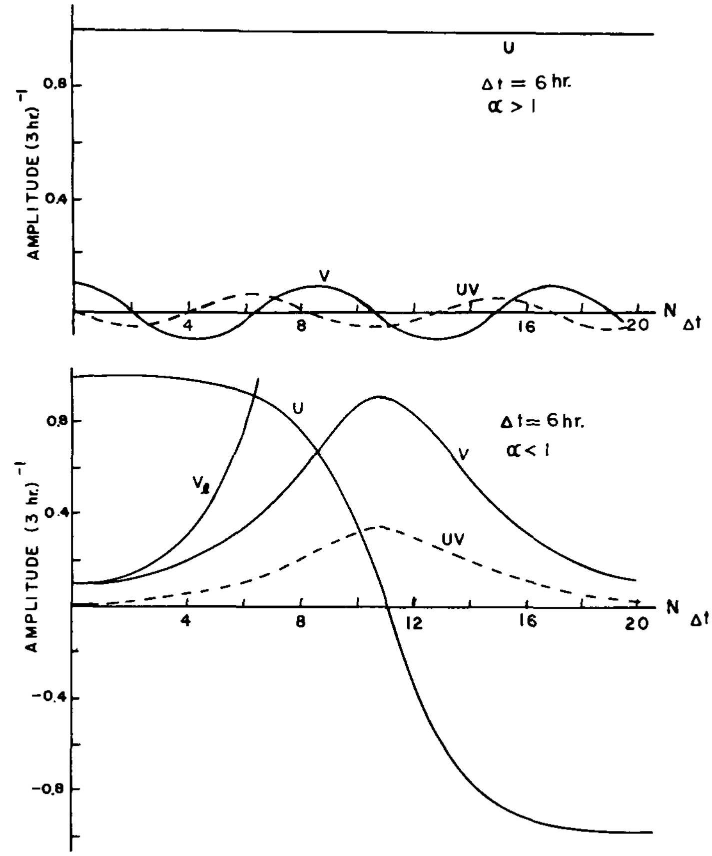

FIGURE 1.—The amplitudes of Model 1 as a function of time for a linearly stable case (top) and a linearly unstable case (bottom). The curve labelled V_l is the time variation of the v component according to the linear analysis.

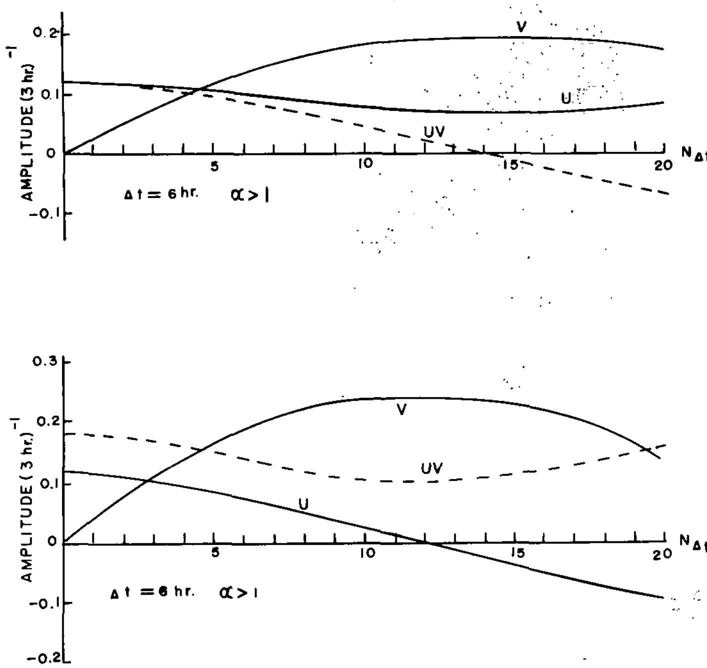

FIGURE 2.—The amplitudes of Model 1 as a function of time for a linearly stable case where perturbation conditions no longer apply.

functions. They may also be solved by simple numerical integration; and it is perhaps somewhat easier to do so, especially if we wish to change parameters and initial conditions to obtain a variety of solutions.

A, in the equations (2) through (4), represents a zonal flow with a sinusoidal profile, F and G being waves superimposed on the zonal flow. Let us then consider a basic zonal flow $A=\overline{A}$ and perturbations $F=F'$, $G=G'$. Then, linearizing equations (3), i.e. neglecting products of perturbation quantities, we have

$$\frac{d\overline{A}}{dt}=0,$$

$$\frac{d^2F'}{dt^2}=-\frac{\alpha^2(\alpha^2-1)}{2(\alpha^2+1)}\overline{A}^2F', \tag{5}$$

with a similar expression for G'. The above equation is of the form $d^2x/dt^2=kx$, and solutions to this equation are exponential if $k>0$ and sinusoidal if $k<0$. Thus for stable oscillations $\alpha^2-1>0$ or since $\alpha\geq 0$, $\alpha>1$; for unstable oscillations $\alpha^2-1<0$ or $\alpha<1$. At this point most analyses of the dynamical equations stop. In this system one is not so limited. One may, in fact, study the nonlinear behavior of the system in conjunction with at worst a simple numerical integration with respect to time.

In figure 1 we present the results of two numerical integrations of equations (3). Time is measured in units of 3 hr., so that if $A=1$, the vorticity of the zonal flow is of the order of f in middle latitudes. The time step used was 2 units or 6 hr. The initial conditions were chosen to be perturbation conditions, that is, so that the linearized equations would be valid initially. The curves are labelled U, v, uv according to the wind components they imply, i.e., $U=A$, $v=F$, $uv=G$.

In the linearly stable case the perturbations oscillate sinusoidally and their amplitudes remain very small. (They do cause slight fluctuations in the zonal component as is necessary for any non-zero perturbation.) The periods of fluctuation are in good agreement with the results of the linear analysis. In the linearly unstable case the perturbations initially grow exponentially, taking energy from the zonal current. The growth does not proceed without limit but is slowed down as the zonal flow becomes weaker, finally ceasing altogether when the zonal flow becomes zero. The perturbations then decay and feed energy back into the zonal flow which now changes sign. The process of growth and decay then repeats itself.

In the linearly stable case A remained practically constant. This is because the perturbations F and G were small initially and always remained small. If, however, one starts an integration where F and G are no longer small relative to A we have the possibility of causing large fluctuations in A. In figure 2 we present the results of numerical integration of equations (3) for a linearly stable case where perturbation conditions no longer apply.

In the first case the initial perturbations are not strong enough to take all the energy from the zonal flow, but do cause a large fluctuation. In the second, the initial perturbations have been increased slightly, and the zonal flow is completely depleted and then reversed in a similar manner to the linearly unstable case.

The above constitutes a minimum system of equations, as devised by Lorenz [5], capable of representing nonlinear barotropic motion. Stability and instability here appear to involve the same process, and the particular motion and development of a perturbation, stable or unstable in a linear sense, is governed by its nonlinear interaction with the basic flow. The only difference between stability and instability is the amplitude of the fluctuations of the various modes of motion. That this amplitude of fluctuation depends on the relative magnitudes of the perturbations and zonal flow illustrates what may be called instability depending on the size of the perturbation.

3. MODEL 2—MOTION ON THE "β-PLANE"

The extension of the Lorenz model to the "β-plane" is quite straightforward. The governing equation for this system is

$$\frac{\partial}{\partial t}(\nabla^2\psi)+\mathbf{k}\times\nabla\psi\cdot\nabla(\nabla^2\psi)+\beta\frac{\partial\psi}{\partial x}=0; \tag{6}$$

and the simplest possible truncation of the system is

$$\nabla^2\psi=A\cos ly+F_1\cos kx+F_2\sin kx+(G_1\cos kx+G_2\sin kx)\sin ly,$$

$$\psi=-\frac{A}{l^2}\cos ly-\frac{F_1}{k^2}\cos kx-\frac{F_2}{k^2}\sin kx-(G_1\cos kx+G_2\sin kx)\frac{\sin ly}{k^2+l^2}, \tag{7}$$

where k, l have the same meaning as in the previous model. Because of the free phase propagation generated by the Rossby term $\beta\ \partial\psi/\partial x$ both the amplitude and phase of each wave must be included.

The harmonic tendency equations resulting from the substitution of the representation (7) into (6) have the following form:

$$\frac{dA}{dt}=\frac{1}{2\alpha(1+\alpha^2)}I_m(FG^*),$$

$$\frac{dF}{dt}=-i\left[\frac{\alpha^3}{2(1+\alpha^2)}AG+\frac{\beta}{k}F\right],$$

$$\frac{dG}{dt}=-i\left[\frac{\alpha^2-1}{\alpha}AF+\frac{\beta\alpha^2}{k(\alpha^2+1)}G\right], \tag{8}$$

where $\alpha=k/l$, $F=F_1+iF_2$, $G=G_1+iG_2$.

For reference, A can be identified with the previous A, F_1 with F, G_2 with $2G$. If $\beta=0$ then F_2 and G_1, being zero initially, would always remain so. Again A may be identified with a zonal flow and F_1, F_2, G_1, G_2 with perturbations on this basic current.

The first step in the discussion of this system will be a linear analysis of the equations (8). As before $A=\bar{A}$, which is large compared to the perturbations F_1', F_2' G_1', G_2'. Neglecting products of perturbations equations (8) become

$$\frac{dF'}{dt}=-i\left[\frac{\alpha^3}{2(1+\alpha^2)}\bar{A}G'+\frac{\beta}{k}F'\right],$$

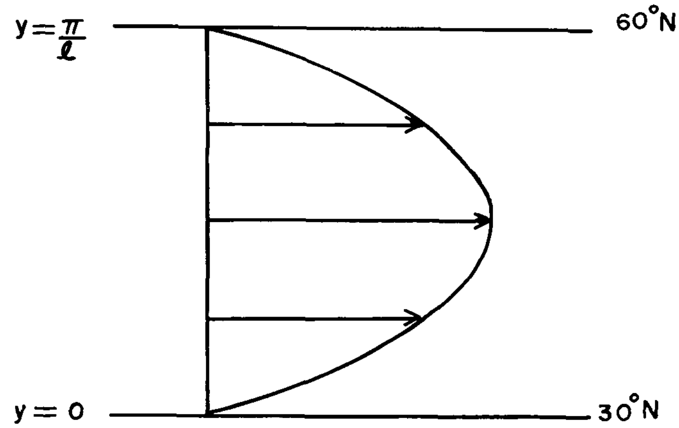

FIGURE 3.—The profile of the zonal current for Model 2.

$$\frac{dG'}{dt}=-i\left[\frac{\alpha^2-1}{\alpha}\bar{A}F'+\frac{\beta\alpha^2}{k(\alpha^2+1)}G'\right], \tag{9}$$

where $F'=F_1'+\mathrm{i}\ F_2'$, $G'=G_1'+\mathrm{i}\ G_2'$.

If we assume perturbations of the form $e^{i\omega t}$, equations (9) yield the following frequency equation

$$\omega=-\frac{\beta}{k}\frac{(2\alpha^2+1)}{(2\alpha^2+2)}\pm\frac{1}{2}\left[\frac{\beta^2}{k^2}\frac{1}{(\alpha^2+1)^2}+\frac{2\alpha^2(\alpha^2-1)}{\alpha^2+1}\bar{A}^2\right]^{1/2}. \tag{10}$$

Thus if α is greater than one, the waves are stable; if α is less than one the necessary and sufficient condition for stability is

$$\bar{A}^2\leq\frac{(\beta/l)^2}{2\alpha^4(1-\alpha^4)}. \tag{11}$$

This stability condition is quadratic in α^4, so that for given values of the zonal flow there will be upper and lower wavelength bonds on the unstable modes.

From equation (11) we find that all wavelengths will be stable if $\bar{A}^2<2\ \beta^2/l^2$, whereas if we consider the known sufficient condition for stability, namely that the gradient of absolute vorticity is of one sign throughout the fluid we find $\bar{A}^2<\beta^2/l^2$. This is reasonable agreement in view of the severe restrictions we have imposed on the flow.

The roots of the frequency equation (10) were computed with the zonal current modelled as illustrated in 3, where the amplitude remains variable. Thus, $Ly=\pi a/3$, so that $l=6/a$; where a is the radius of earth. Time is measured in units of approximately 18 hours, so that $f=6$ and $\beta=f/a=6/a$. These parameters were also used in the numerical integration to be described later.

Figure 4 shows the real and imaginary parts of the frequency as a function of α for various amplitudes of the basic zonal flow. The linear analysis of this system as presented in figure 4 indicates the following general features:

a) Very long waves $\alpha=Ly/Lx\ll1$ are stabilized by the inclusion of the Rossby term. The two phase speeds are very widely different, being determined primarily by the β effect, as may be seen by considering equation (10) when $\alpha\ll1$.

b) Short waves $\alpha=Ly/Lx>1$ are stable as before, and their phase speeds are determined primarily by the zonal current.

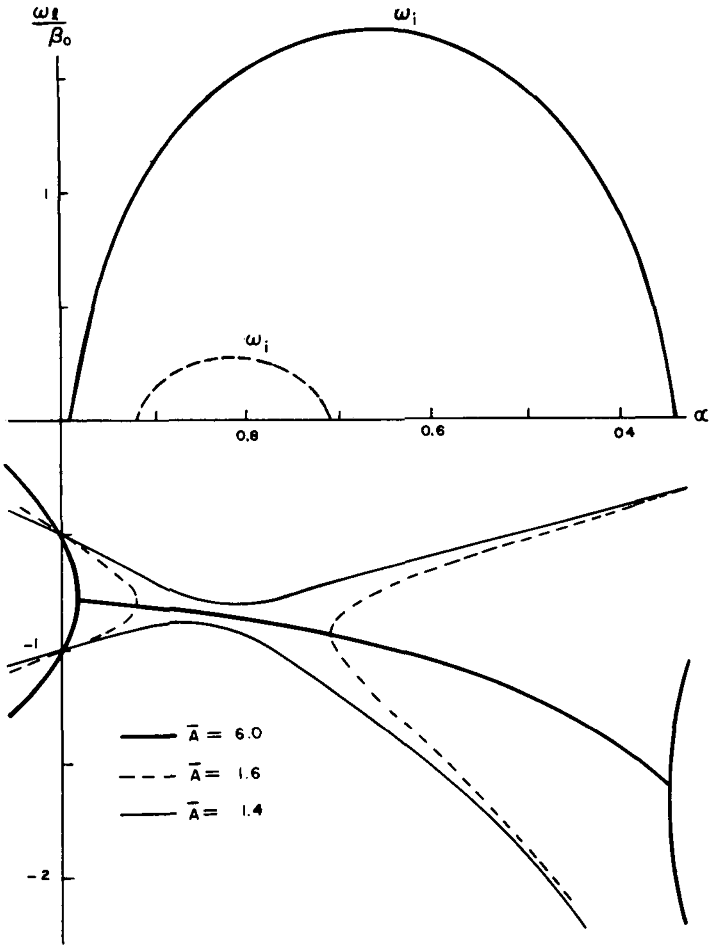

FIGURE 4.—The real part and positive imaginary part of the frequency as a function of the ratio of fundamental wavelengths for different values of the zonal wind amplitude (Model 2).

c) Intermediate wavelengths may be unstable if the zonal current is strong enough.

This picture is consistent with the barotropic analysis of Kuo [4], although instability is more difficult to achieve in the sense that greater zonal wind shear is required. These results are also similar to those obtained by Wiin-Nielsen [10] in the sense that the β-term tends to stabilize the longer wavelengths and that intermediate wavelengths may be unstable if the zonal flow is strong enough. The system of equations (8) may be shown to have solutions which are elliptic functions of time (Platzman [7]), but again by simple numerical integration we may study the nonlinear properties of the model.

In the following three cases the initial conditions are the same and correspond to perturbation conditions, specifically $A=-6$, $F_1=F_2=G_1=G_2=0.1$ at $t=0$. The mean square vorticity of each component (which in this system is proportional to the kinetic energy) as well as the phase angle of the two waves plotted as functions of time in figure 5. Again, A is referred to as "U," F_1, F_2 as the function "v" wave, and G_1, G_2 as the "uv" wave.

CASE I. α=1.5 (LINEARLY STABLE)

The two waves interchange energy and the zonal current undergoes only very slight changes (0.001 percent and thus is not plotted). The average angular phase speeds

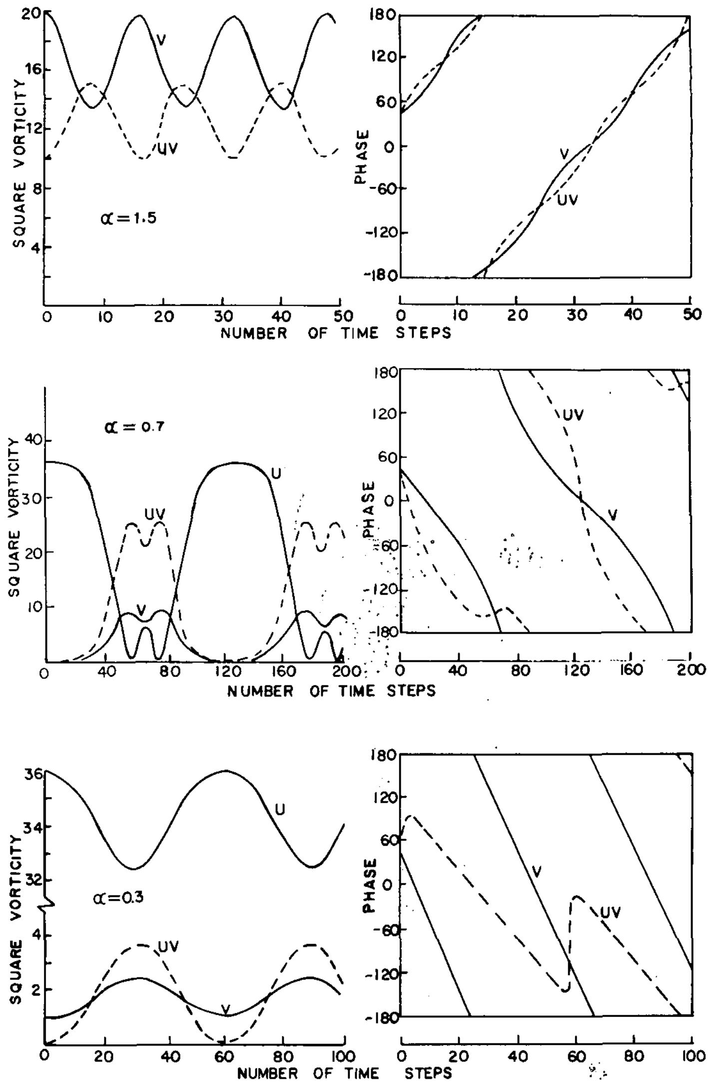

FIGURE 5.—The mean square vorticity and phase angle of the components of Model 2 as a function of time for a short wave stable case (upper), a long wave stable case (lower), and an unstable case (center). The time step is approximately 1 hour.

of the waves correspond very closely to the results of the linear analysis. The two waves do vary their relative positions periodically but with small amplitude.

CASE II. α=0.7 (LINEARLY UNSTABLE)

In this case the waves are linearly unstable and start to grow at once, extracting energy from the zonal current. Soon all of the energy is removed from the zonal current which changes sign for a short period, and the perturbations have reached their maximum intensity. The perturbations then decay, feeding their energy back to the zonal current which climbs back to its original value. The "uv" wave lags the "v" wave when the zonal current is decreasing westerly and shifts to be leading when the zonal current is increasing westerly.

CASE III. α=0.3 (LINEARLY STABLE)

The zonal current undergoes a weak sinusoidal fluctuation of about 10 percent of its amplitude. The perturbations oscillate sinusoidally as well, both being out of phase with the zonal current. The phase progression of the waves

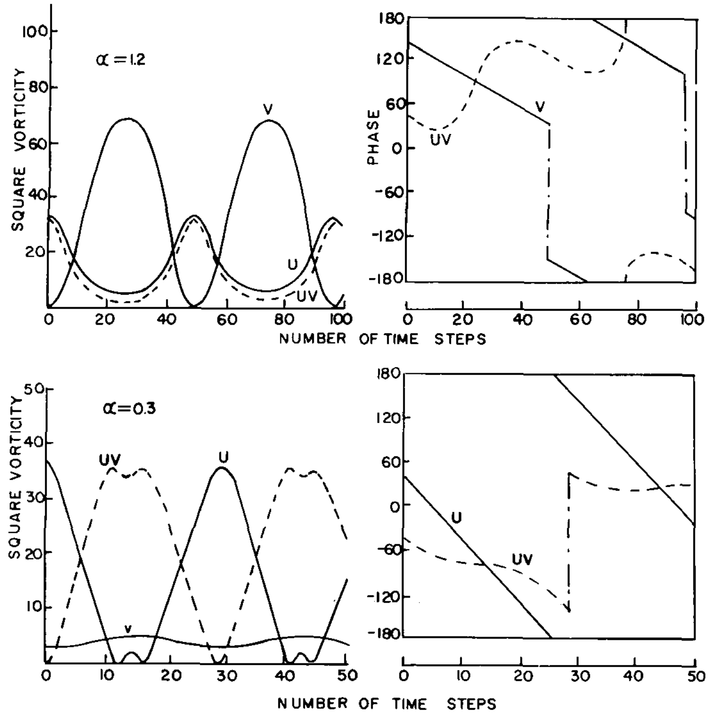

FIGURE 6.—The mean square vorticity and phase angles of the components of Model 2 as a function of time; where perturbation conditions do not apply. The upper diagram is for a short wave case, the lower, a long wave case; both cases are linearly stable. The time step is again approximately 1 hour.

is quite different from Case I. Here the waves are retrogressing and have phase variations which are similar to Case II. However, because the average phase speeds of the two waves are widely different, the amount of time spent in one phase configuration is small.

As in Model 1 discussed previously we can cause large fluctuations in a stable zonal current merely by having large enough perturbations initially. Figure 6 illustrates this for both long wave and short wave cases.

The question now is what physical significance or conclusions can be drawn from this simple nonlinear system? Firstly, the nonlinear process in operation here is horizontal momentum transport and convergence. The covariance of the convergence of momentum transport and the zonal wind is, of course, the measure of the energy conversion from eddy kinetic energy to zonal kinetic energy. The horizontal momentum transport in this system is given by

$$M_T=\frac{1}{Lx}\int_0^{Lx} u'v'dx=-\frac{1}{2\alpha(1+\alpha^2)}\frac{1}{l^2}(F_2G_1-F_1G_2)\cos ly. \tag{12}$$

The connection with the rate of change of zonal kinetic energy is apparent because

$$\frac{dA}{dt}=\frac{1}{2\alpha(1+\alpha^2)}(F_2G_1-F_1G_2).$$

Now if we let

$$F_v=(F_1^2+F_2^2)^{1/2} \text{ and } \phi_v=\tan^{-1}\frac{F_2}{F_1};$$

$$G_{uv}=(G_1^2+G_2^2)^{1/2} \text{ and } \phi_{uv}=\tan^{-1}\frac{G_2}{G_1};$$

then (12) becomes

$$M_T=-\frac{1}{2\alpha(1+\alpha^2)}\frac{1}{l^2}F_vG_{uv}\sin(\phi_v-\phi_{uv})\cos ly. \tag{13}$$

Thus, south of the wind maximum we have southward transport of westerly momentum if $\phi_v>\phi_{uv}$. Because of the simplicity of the system the momentum convergence has the same profile as the zonal current so that no splitting or north-south motion of the wind maximum can be produced. The important thing to note is that the momentum transport depends on the difference in phase of the "v" and "uv" waves. From this consideration the following physical picture presents itself. The two waves ("v," "uv") are moving in an east-west direction. Even if they are in phase at one particular time, because of their different scales (resulting in different Rossby phase speeds), they become out of phase at a later time; and transport and converge momentum changing the zonal current. This change in the zonal current produces a change in the phase speeds of the two waves and an oscillation has been started. The details of the motion then depend on how much of an effect the perturbations have on the zonal current. If they have little effect then the oscillation is weak and we may consider the system as stable and if they have a large effect we may consider the system as unstable. The effect must be on the zonal current, and it is not sufficient just to have large momentum transports. In fact, this is the reason why the inclusion of the Rossby term stabilizes the long waves. Because of the large difference in Rossby phase speeds of the "v" and "uv" waves, and although for this scale of motion the momentum transport is large, no significant energy conversion takes place because there is not enough time. In other words, although the amplitude of the momentum transport is large, its time frequency is also large; so that it changes direction before it performs a significant energy conversion.

4. MODEL 3—MOTION ON A SPHERICAL SURFACE

The starting point for this model is the same as for Model 2, i.e., the barotropic vorticity equation, but without the "β-plane" approximation. This model does have an added complexity since for particular components representing the zonal flow it is possible to have a zonal wind which is non-zero when averaged in a north-south direction.

The spectral equation describing barotropic flow in terms of spherical harmonics has been given by Platzman [7] and may be expressed as

$$-C_\gamma\frac{d\psi_\gamma}{dt}=i\sum_\alpha\sum_\beta\psi_\alpha\psi_\beta H_{\alpha\gamma\beta}-i2\Omega m_\gamma\psi_\gamma, \tag{14}$$

where

$$\psi=a^2\sum_\gamma\psi_\gamma Y_\gamma;\ Y_\gamma=P_\gamma e^{im_\gamma\lambda};\ C_\gamma=n_\gamma(n_\gamma+1).$$

In this equation $H_{\alpha\gamma\beta}$ is the interaction matrix and is given by

$$H_{\alpha\gamma\beta}=\frac{C_\alpha-C_\beta}{2}\int_0^\pi P_\gamma\left(m_\alpha P_\alpha\frac{dP_\beta}{d\theta}-m_\beta P_\beta\frac{dP_\alpha}{d\theta}\right)d\theta;$$

if $m_\alpha+m_\beta=m_\gamma$, $H_{\alpha\gamma\beta}=0$;

otherwise, where $P_\gamma = P_{n\gamma}^{m\gamma}$ is the normalized Legendre function of the first kind.

The simplest truncation which produces nonlinear exchange is a three component one. Thus, one component represents the zonal flow, ψ_n^0, and the other two represent perturbation of longitudinal wave number m; ψ_s^m and ψ_k^m where $s \neq k$. This corresponds to Platzman's class $L3$ (Platzman [7]). So that in (14) α, β, γ can take on the set of values $(0, n)$, $(\pm m, k)$, $(\pm m, s)$.

Performing the indicated summation and using the symmetry and redundancy properties of the H's (Silberman [9], Platzman [7]), the spectral equations take the form

$$-C_n \frac{d\psi_n^0}{dt} = i(\psi_k^m \psi_s^{*m} - \psi_s^m \psi_k^{*m})\, m(C_k - C_s)\xi,$$

$$-C_s \frac{d\psi_s^m}{dt} = im\psi_n^0 (\psi_k^m (C_n - C_k)\xi - \psi_s^m (C_n - C_s)\alpha_s) - i2\Omega m \psi_s^m,$$

$$-C_k \frac{d\psi_k^m}{dt} = im\psi_n^0 (\psi_s^m (C_n - C_s)\xi - \psi_k^m (C_n - C_k)\alpha_k) - i2\Omega m \psi_k^m, \tag{15}$$

where

$$\xi = -\int_0^\pi P_k^m P_s^m \frac{dP_n^0}{d\theta}\, d\theta,$$

$$\alpha_s = \int_0^\pi (P_s^m)^2 \frac{dP_n^0}{d\theta}\, d\theta,$$

$$\alpha_k = \int_0^\pi (P_k^m)^2 \frac{dP_n^0}{d\theta}\, d\theta.$$

The horizontal mean of kinetic energy in this system is

$$\overline{E} = \frac{a^2}{4}(C_n \psi_n^{0 2} + 2C_k \psi_k^m \psi_k^{*m} + 2C_s \psi_s^m \psi_s^{*m}),$$

$$= \frac{a^2}{4}(E_n^0 + E_k^m + E_s^m); \tag{16}$$

while the mean square vorticity is given by

$$\overline{V}^2 = \tfrac{1}{2}(C_n^2 \psi_n^{0 2} + 2C_k^2 \psi_k^m + \psi_k^{*m} + 2C_s^2 \psi_s^m \psi_s^{*m}),$$

$$= \tfrac{1}{2}(\overline{V}_n^{2 0} + \overline{V}_k^{2 m} + \overline{V}_s^{2 m}). \tag{17}$$

Using the tendency equations (15) the energy and vorticity exchange in this system is described by

$$\frac{dE_n^0}{dt} = (C_s - C_k)Q, \qquad \frac{d\overline{V}_n^{2 0}}{dt} = C_n(C_s - C_k)Q,$$

$$\frac{dE_k^m}{dt} = (C_n - C_s)Q, \qquad \frac{d\overline{V}_k^{2 m}}{dt} = C_k(C_n - C_s)Q,$$

$$\frac{dE_s^m}{dt} = (C_k - C_n)Q, \qquad \frac{d\overline{V}_s^{2 m}}{dt} = C_s(C_k - C_n)Q, \tag{18}$$

where

$$Q = 2i\psi_n^0 (\psi_k^m \psi_s^{*m} - \psi_k^{*m} \psi_s^m) m\xi.$$

This is an expression of the Fjørtoft blocking theorem (Fjørtoft [5]); and it can be seen that

$$\frac{d\overline{E}}{dt} = \frac{d\overline{V}^2}{dt} = 0.$$

Linearizing the equations and considering time variations of the form e^{-imct} the following frequency equation is obtained:

$$c^2 + c(\gamma_s + \gamma_k) + \gamma_s \gamma_k - \delta_s \delta_k = 0, \tag{19}$$

where

$$\gamma_s = \frac{2\Omega + \psi_n^0 \alpha_s (C_n - C_s)}{C_s}, \qquad \delta_s = \frac{\psi_n^0 (C_n - C_s)}{C_k}\xi,$$

$$\gamma_k = \frac{2\Omega + \psi_n^0 \alpha_k (C_n - C_k)}{C_k}, \qquad \delta_k = \frac{\psi_n^0 (C_n - C_k)}{C_s}\xi.$$

Thus the necessary and sufficient condition for stability is that

$$\delta_s \delta_k \geq -\frac{1}{4}(\gamma_s - \gamma_k)^2, \tag{20}$$

and the phase speeds are given by

$$C = -\frac{\gamma_s + \gamma_k}{2} \pm \frac{1}{2}\{(\gamma_s - \gamma_k)^2 + 4\delta_s \delta_k\}^{1/2}. \tag{21}$$

The above result is identical to that obtained by Platzman [7].

The physical meaning of the parameters are the following: $-\gamma_s$, $-\gamma_k$ are the Rossby phase speeds (or convective phase speeds), δ_s, δ_k are nonlinear phase speeds (depending on momentum transport). Now, $\delta_s \delta_k$ is given by

$$\delta_s \delta_k = \frac{\psi_n^{0 2} \xi^2}{C_s C_k}(C_n - C_k)(C_n - C_s) \tag{22}$$

and from (20) if $\delta_s \delta_k \geq 0$ then waves are stable. Thus it follows that unless the zonal wave scale is intermediate to the other two, and $\xi \neq 0$, the waves are stable. Since the equations are symmetric in s, k let $s > k$. Then waves are stable unless

$$k < n < s \text{ and } \xi \neq 0.$$

Now, $\xi \neq 0$ only if $s + k + n = $ odd and $|s - n| k < s + n$ (Silberman [9]). So that the number of possible unstable modes is quite restricted. The combinations of (s, k) which can be unstable for a given zonal component are given in table 1.

As indicated previously this model has the added feature that a purely convective phase speed (one that does not require energy exchange) involving the zonal component is possible. This is measured by the parameters α_s and α_k. It is instructive therefore to compare the case where $\alpha_s = \alpha_k = 0$ with the stability criterion of Model 2. Since

$$\alpha_s = \int_0^\pi (P_s^m)^2 \frac{dP_n^0}{d\theta}\, d\theta,$$

TABLE 1.—*Possible unstable modes for a given zonal component*

n	1	2	3	4	5	6
(s, k) values	None	None	2, 4	2, 5 3, 6	2, 6 3, 7 4, 6 4, 8	2, 7 3, 8 4, 7 4, 7 5, 8 5, 10

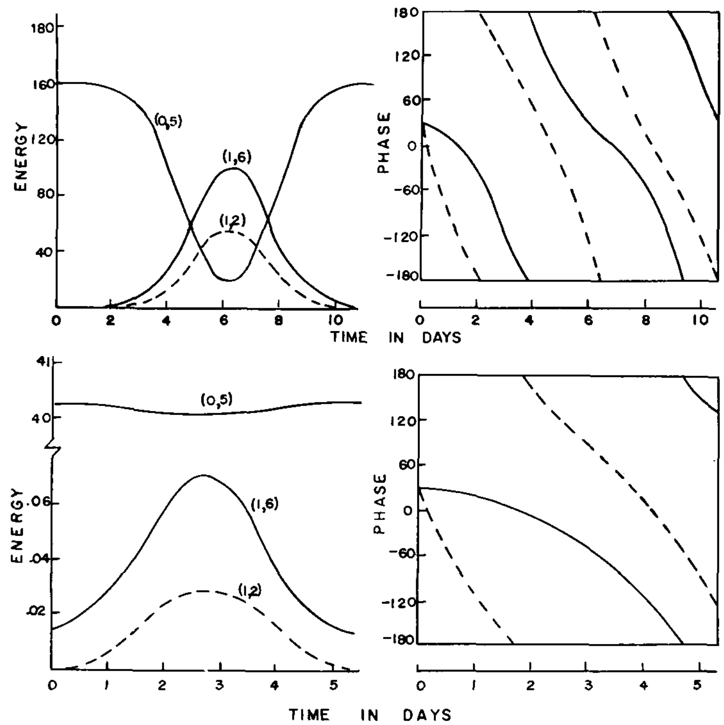

FIGURE 7.—The kinetic energy (relative units) and phase angles of the components of Model 3 as a function of time. The upper figure is for an initial value of the zonal component which is unstable; the lower for a stable initial value.

it follows that P_n^0 must be an even function of latitude, i.e., n=even; which corresponds to an odd zonal wind field. Assuming that $k<n<s$ and $\xi\neq 0$ the condition for instability is then

$$\psi_n^{0^2}>\frac{(\gamma_s-\gamma_k)^2C_sC_k}{4\xi^2(C_n-C_k)\,(C_s-C_n)} \tag{23}$$

where $\gamma_s=2\Omega/C_s$, $\gamma_k=2\Omega/C_k$, i.e. the Rossby phase speeds for a zero zonal wind. Now, the wavelengths of the components are defined as $L_n^2=a^2/C_n$, etc., where a is the radius of the earth so that (23) becomes

$$\psi_n^{0^2}>\frac{(\gamma_s-\gamma_k)^2}{4\xi^2\left(\left(\frac{L_k}{L_n}\right)^2-1\right)\left(1-\left(\frac{L_s}{L_n}\right)^2\right)}. \tag{24}$$

It may be shown that the stability criterion in Model 2 can be written as

$$\left(\frac{A}{l}\right)^2>\frac{(\gamma_v-\gamma_k)^2}{2\left(\left(\frac{L_v}{L_u}\right)^2-1\right)\left(1-\left(\frac{L_{uv}}{L_u}\right)^2\right)} \tag{25}$$

where γ_v, γ_{uv} are the Rossby speeds, and L_u, L_v, L_{uv} are the wavelengths of the components. Thus Model 2 and this case of Model 3 are physically equivalent. Model 2 could be made completely physically equivalent to Model 3 if a constant zonal wind was added to the sinusoidal profile since the β-plane approximation was employed.

As far as the application of the results of the stability analysis is concerned it has been shown by Merilees [6] in an analysis of the stream field at 500 mb. for September 1957 that no component ever had an amplitude during the month which was unstable with respect to perturbations of this kind. This would suggest that if motion of this type were to be observed it would be of the stable type (which may, of course, involve large fluctuations of the zonal component if the perturbations are large enough).

The integrations of equation (15) (because of their similarity to equation (8)) produce essentially the same results as Model 2. In figure 7 we show the results of two cases of numerical integration of equation (15). In the first case (upper) the initial value of ψ_n^0 is in the unstable regime, the other (lower) is for a stable initial value of ψ_n^0. In both cases the initial values of the perturbations were one hundredth of the initial values of ψ_n^0. Thus the same interpretation of the stability criterion and the numerical integrations in terms of horizontal momentum transport and convergence as was formulated in Model 2, apply to Model 3.

5. SUMMARY AND COMMENTS

Three simple models of barotropic flow have been studied by use of the spectral method. The results of these studies are consistent with previous linear analyses of barotropic flow but also provide a physical interpretation of barotropic instability in terms of the frequency of momentum transport and convergence. It may well be argued that the severe truncation of the representation in these models limits their applicability in detail to atmospheric flow. However, the mechanism of energy exchange (i.e. momentum convergence) will be the same no matter how complete or incomplete the representation. So, if we limit ourselves to understanding this exchange process then we are most probably on safe ground.

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